ECON 7010 - Macroeconomics I

Fall 2015

Notes for Lectures #8 and #9

Today:

- Overlapping Generations Models
  - Endowment economy
  - Production economy

Overlapping Generations (OG) Models
(Note: we'll solve OG models in two ways:)
Allocations:

Decentralized

- individual optimization (we did this with dynamic programming)
- consistency requirement
  1. Markets clear (non-strategic)
  2. Nash equilibirum (strategic)

→ Need optimization and consistency requirement for competitive equilibrium

Centralized (Planner) (note, we did this way w/ the growth model)

- Good:
  - Easy to solve optimization problem
- Bad:
  - May not accord with decentralized allocation (Fundamental Welfare Theorem may not hold)
  - Care about how allocations are decentralized (institutional design)

OG Presentation

- Shell's paper (1971) (post-Samuelson, 1958) - endowment economy
- Simple production economy
- Gale (1973) endowment economy (NOTE: these first three papers don't have capital - dynamics are solely result of generational dynamics)

Shell's Example: Environment:

- Time: $t = 1, 2, ..., \infty$
- Demographics:
  - N_t agents born in period t
  - All agents live 2 periods
    - So each period so some old and young
  - $N_t = \gamma N_{t-1}$, γ gives rate of pop growth
  - N_0 given

- Shell assumes: $N_t = 1, \forall t$; $\gamma = 1$ (i.e, no pop growth); $N_0 = 1$
- At period 1, have the N_1 young and N_0 old (young of generation 1, old of generation 0)

- Competitive market (note, we treat each generation as one agent, but really a continuum of identical agents, all of whom are price takers)
- Preferences:
  - Generation t agent lives 2 periods
  - Lifetime utility $= u(c_t^t, c_{t+1}^t)$, where superscript is for the generation, subscript for the period
  - Shell assumes: $u(c_t^t, c_{t+1}^t) = c_t^t + c_{t+1}^t$ (i.e., goods are perfectly substitutable across periods)
- Endowment: $e_t = (e_t^t, e_{t+1}^t)$: endowment vector over lifetime of generation t
  - For Shell, $e_t = (1, 1), \forall t$ - flat endowment over lifetime
- Technology:
  - No production
  - No storage (i.e., no means of taking stuff to the next period)
  - $\Rightarrow$ consume everything in one period
- Budget constraint:
  - $p_t \equiv$ price of goods in period t relative to period one goods
    * p_1 is the numeraire; $p_1 = 1$
  - People of all generations get together to make transactions at the beginning of time
    * $\Rightarrow$ Markets opening and closing sequentially is not a source of market failure
  - Generation t agent's budget constraint: $p_t c_t^t + p_{t+1} c_{t+1}^t = p_t e_t^t + p_{t+1} e_{t+1}^t = p_t + p_{t+1}$
- Competitive equilibrium
  - Agents act optimally (i.e., max utility subject to budget constraint)
  - Markets clear (i.e, supply = demand)

$$\underline{\text{Demand}} \qquad\qquad\qquad \underline{\text{Supply}}$$

$$\underbrace{c_t^{t-1} + c_t^t}_{\text{period } t \text{ demand}} \quad = \quad \underbrace{e_t^{t-1} + e_t^t}_{\text{period } t \text{ supply}}, \quad t = 1, ..., \infty$$

  - $\Rightarrow \{p_t\}_{t=1}^{\infty}, \{c_1^0, (c_t^t, c_{t+1}^t)\}_{t=1}^{\infty}$
  - Note: Demand and supply are supply in period t are from the two generations alive at time t - those in the $t-1$ generation (the old at time t) and those in the t generation (the young at time t)

Model Solution

- Individual optimization
  - Lifetime budget constraint for generation t agent:
    * $p_t c_t^t + p_{t+1} c_{t+1}^t = p_t + p_{t+1}$
    * $\Rightarrow p_{t+1} c_{t+1}^t = p_t + p_{t+1} - p_t c_t^t$

    - $\Rightarrow c_{t+1}^t = \frac{p_t(1-c_t^t)+p_{t+1}}{p_{t+1}}$
    - $\Rightarrow c_{t+1}^t = (\underbrace{p_t}_{\text{Price sold at}} \underbrace{((1-c_t^t)}_{\text{Period } t \text{ endow giv up}} / \underbrace{p_{t+1}}_{\text{Price bought at}}) + 1$
  - DRAW graph for generation t. Axes are consumption in t and $t+1$. Note that slope of budget constraint $= -\left(\frac{p_t}{p_{t+1}}\right)$. Note that indifference curves are straight lines with slope = -1.
  - Indifference curves tell how you want to trade
  - Budget constraint defines how can trade
  - Agents make best decisions given preferences and constraints

- Competitive Equilibrium
  - $p_t = 1, \forall t$ (should have been able to guess this from example, since # born, endowment, and preferences, the same in all periods - if $p_t \neq 1$ then corner sol'n (consume all when old or young), but this not possible with initial old- nothing for them to trade for/not willing to give up endowment before die)
  - $(c_t^t, c_{t+1}^t) = (e_t^t, e_{t+1}^t) = (1,1)$, and $e_1^0 = 1$ - this is Autarky (no trade), it's the only equilibrium in this model
  - Since generations are only together one period, and you want to consume all before you die, the "old" generation has nothing to trade for (remember, there is only one good)
  - Is the competitive equilibrium Pareto optimal?? (note, this is Shell's big question)
    - Is society better off with some other allocation?
    - Is there another <u>feasible</u> allocation that would make everyone at least as well off and someone better off?
    - Remember that an equilibrium is characterized by an allocation and prices
    - DRAW box with period and generation, showing consumption in each period. Point out why that first generation won't give up any of his endowment and so can't trade and this unravels the whole thing
    - The CE is optimal, if T is finite
      - No better allocation w/ T generations b/c moving consumption leaves someone worse off
    - CE not optimal if T is infinite
      - Shift from young to old, passing chocolate "down" a generation
      - This gives generation zero $c_1^0 = 2$, which is better than before
      - All other generations are just was well off as before b/c $u(c_t^t, c_{t+1}^t) = c_t^t + c_{t+1}^t$, consumption in one period is a perfect substitute for consumption in another. So utility the same whether each generation consumes $(1,1)$ or $(0,2)$.
      - No one is worse off because the passing down never ends ($T = \infty$).
      - $\Rightarrow$ CE not Pareto optimal

- Important ingredients for CE not being Pareto optimal:
  - OG vs. infinite horizon (Need lifetimes of agents to end, need overlapping to shift)
  - T is infinite (else unwinds b/c someone worse off)
  - No discounting; goods perfect substitutes across periods
  - $N_t = 1, \forall t$ (no pop growth)
  - no storage, no production
  - (NOTE: Shell points out that it's not incomplete markets that cause the 1st welfare theorem not to hold - it's the double infinity of goods and generations - the CE is not optimal even with a period 0 claims market)

- Gale paper (with production) will take on: MRS at equilibrium (i.e., what compensation scheme can make this work?) and γ (i.e., what compensation is feasible given changing population).

How can society achieve the (0,2) allocation?

1. Social security: tax/transfer (tax young, give money to old)
2. Money (specifically, fiat money)
   - Intrinsically useless, only useful for purposes of exchange
   - Not gov't backed
   - Not in $u(\cdot)$ (if value to money, then won't work)
   - Not in $f(\cdot)$ (production function)
   - Does fiat money have value in equilibrium? It depends - value of money depends upon self-reinforcing beliefs
   - How equilibrium with money works:
     - Let $\pi_t \equiv \frac{\#\ \$\text{s}}{good}$, $t = 1, ..., \infty$ (=dollar price tag of goods, in dollars so a nominal variable)
     - Budget constraint for generation t (w/ money and a sequence of markets (no meeting at beginning of time)
       * Period t budget constraint: $\underbrace{\frac{m_t}{\pi_t}}_{\text{real money demand}} = \underbrace{(e_t^t - c_t^t)}_{\text{real savings= endow-cons}}$
       * m_t= nominal money demand of t generation at young age (i.e.,s how many dollars they demand)
       * NOTE: demand for money by the old = 0, they can't wait eat it and don't value future consumption
       * Period $t+1$ budget constraint: $c_{t+1}^t = e_{t+1}^t + \frac{m_t}{\pi_{t+1}}$ (notice no uncertainty - you know the value of money in the future)
     - Where does money come from?
       * Chocolate wrapper from initial old: $\Rightarrow$ money supply = 1 wrapper
       * No other money creation (for now)
       * Note: if money in $u(\cdot)$, then initial old will eat it and this won't work

Competitive equilibria with money: (lots of equilibria, but focus on steady state prices and allocations: $\pi_t = \pi^*, c_t \equiv (c_t^t, c_{t+1}^t) = c^*$)

1. $\pi^* = 1, c^* = (0,2), c_1^0 = 2$
   - (C.E. because on budget line and slope = indiff curve (could have any between (1,1) and (0,2), but initial old don't value dollar so want $c_1^0 = 2$ and thus we have $c^* = (0,2)$ for all other generations)
   - Pareto optimal equilibrium (in infinite) supported b/c money has value here (no person in generation T gets stuck holding the bag)
   - Value of money come from the idea that money has value
     - People think that those in the future will give them something for it
     - π_{t+1} has an expected value (it's known) - we will talk more about rational expectations later...
     - Need infinite time so not one stuck "holding the bag".
2. $\pi^* = 0, c^* = (1,1), c_1^0 = 1$

  - Autarky - no trade, money has no value
  - Not Pareto optimal

- There may be other steady states, but we look at only 1) and 2).
- In order for money to have value, we need an infinite horizon
- Note that autarky should be a possible equilibrium in any model with money - else you have assumed money into the model

OG w/ Production (Note: now we start extending the assumptions of Shell to see if they matter)

- Model
  - Demographics: $N_t = N, \forall t$ (zero pop growth), live 2 periods
  - Time: $t = 1, 2, 3...$
  - Preferences: Generation t's utility: $u(c_{t+1}) - g(n_t)$
    * Agents work in first period of life, consume in the second
    * c_{t+1}= consumption in period $t+1$ by generation t
    * n_t= labor supply of generation t in period t
    * The above preferences are for a representative agent - meaning the i's are implicit. So don't get confused and forget that when you are looking at market clearing.
  - Endowment:
    * 1 unit of time to supply when young, $n_t \in [0, 1]$
    * Note: $u' > 0, u'' < 0, g' > 0, g'' > 0$
    * Draw axes with n_t and c_{t+1} and indifference curves
    * No consumption in the first period (young age)
  - Technology (for now):
    * No firms
    * No shocks
    * No capital
    * $y_t = n_t \rightarrow$ can do this without loss of generality ($y_t = f(n_t), f' > 0, f'' < 0$)
- Planner's problem (a static problem (b/c all generations same, planner chooses allocations) $\rightarrow$ solution is a scalar)
  - $c_t = y_t = n_t$ (it's really $Nc_t = Nn_t = Ny_t$, but the N's cancel b/c zero pop growth)
  - $\Rightarrow c^{**} = n^{**} = y^{**}$ (** denotes the sol'n to the planner's problem)
  - Problem is: $\max_{n_t} u(c_t) - g(n_t)$ (max utility for period t b/c all the same) (equal treatment principle implicitly here)
  - At optimum, $c_t = n_t$ and $c_t = c_{t+1}$ (first is b/c resource constraint, second is b/c agents all the same)
  - $n^{**} = \arg\max_{0 \leq n \leq 1} u(n) - g(n)$
    * $\Rightarrow u'(n^{**}) = g'(n^{**})$
    * To get an interior solution, assume: $u'(0) > g'(0)$ and $g'(1) > u'(1)$ (meaning that the slope of the utility function near zero consumption is steeper than the slope of the disutility of labor function near zero labor supply (and the opposite at 1))
    * DRAW graphs with concave u and convex g functions

    * DRAW graph with c and n on axis and indiff curve tangent to production possibilities frontier (=45 degree line given tech - $y_t = n_t$) (NOTE that we get interior solution by assumption on preferences)

- Competitive Equilibrium (a dynamic problem - solution is a sequence of prices and a sequence of employment levels)
  - Representative Genration t agent solves:
    * $\max_{c_{t+1},n_t} u(c_{t+1}) - g(n_t) = \max_{n_t} u(\rho_t n_t) - g(n_t)$
      · subject to: $c_{t+1} = \frac{\pi_t}{\pi_{t+1}} n_t = \rho_t n_t, where \rho_t = \frac{\pi_t}{\pi_{t+1}}$ (ρ_t is a real wage (or real interest rate))
      · $n_t = y_t \to$ output technology
      · $\pi_t n_t \equiv$ nominal income ($)
      · FOC: $\frac{\partial U}{\partial n_t} : \rho_t u'(\rho_t n_t) - g'(n_t) = 0$
      · $\Rightarrow \rho_t u'(\rho_t n_t) = g'(n_t)$
  - Market clearing:
    * Supply = demand in the goods market: $c_t = y_t$
    * Supply = demand in the money market: $\pi_t c_t = M$
  - Equilibrium= $\{\pi_t, n_t\}_{t=1}^{\infty}$
    * Remember: all equilibria are characterized by a price and an allocation
    * Sequence $\{\pi_t, n_t\}_{t=1}^{\infty}$ must satisfy:
      · individual optimization: $\frac{\pi_t}{\pi_{t+1}} u'(\frac{\pi_t}{\pi_{t+1}} n_t) = g'(n_t)$
      · market clearing (for goods, only need to show that $n-1$ of n markets clear)
      · Showing market clearing condition:
      · $\frac{NM}{\pi_t} = N n_t \Rightarrow \frac{M}{\pi_t} = n_t, \quad t = 1, 2, ...,$
      · Where $\pi_t \equiv$ money price of goods = dollars per good
      · $M_t = M, \forall t \equiv$ the money supply per capita
      · Recall, $\pi_t = \frac{\#\$}{\text{good}}$, $M = \$ \Rightarrow \frac{M}{\pi} = \frac{\$}{\frac{\#\$}{\text{good}}} = \#$ goods
  - Perfect foresight solution: (agents see future prices exactly)
    * Only relative prices matter for n_t
      · This is directly evident from $c_t = \frac{\pi_t}{\pi_{t+1}} n_t$ (the B.C.), so if you multiply both π_t and π_{t+1} by the same number, it cancels
    * All agents are the same, but the market is competitive (i.e., all agents are price takers)
    * Walras' Law implies that if markets clear, we only need to look at $n-1$ markets to see this
      · So here, money supply=money demand $\Leftrightarrow$ good market clears (since we only have these two markets)
  - Walras' Law (and aside)
    * Showing why Walras' Law means that you only have to show that N-1 of N markets clear. That is N-1 clears, you know the Nth does as well.
    * Example: OG model with production and money:
      · 2 markets: goods and money
      · Assume the money market clears: $\underbrace{\pi_t c_t}_{\text{demand}} = \underbrace{M}_{\text{supply}}, \forall\, t$
      · $\implies \pi_{t+1} c_{t+1} = M$
      · $\implies \frac{\pi_t}{\pi_{t+1}} = \frac{\frac{M}{c_t}}{\frac{M}{c_{t+1}}} \implies \frac{\pi_t}{\pi_{t+1}} = \frac{c_{t+1}}{c_t}$
      · Now, put this into the budget constraint:
      · B.C.: $c_{t+1} = \frac{\pi_t}{\pi_{t+1}} y_t$

- $\implies c_{t+1} = \frac{c_{t+1}}{c_t} y_t \implies c_t = y_t$
- But $c_t = y_t$ is exactly the goods market cleaning condition.
- Thus we just proved Walras' Law holds here - given that the money market cleared, we can show the the goods market clears as well, because the consumer's budget constraint is satisfied as part of individual optimization.

– Procedure to solve for the C.E.:

1. Solve for necessary conditions of individual optimization problem (i.e., FOC)
   * $\rho_t u'(\rho_t n_t) = g'(n_t)$
2. Solve for the market clearing condition - find p's in terms of n's (nominal in terms of real variables)
   * $\pi_t n_t = M \implies \pi_t = \frac{M}{n_t}$
3. Substitute market clearing conditions into individual optimization conditions (gets rid of $\{\pi_t\}_{t=1}^{\infty}$ to solve for $\{n_t\}_{t=1}^{\infty}$)
   * $\Rightarrow \rho_t = \frac{\pi_t}{\pi_{t+1}} = \underbrace{\left(\frac{M}{n_t}\right)}_{\pi_t} * \underbrace{\left(\frac{n_{t+1}}{M}\right)}_{1/\pi_{t+1}} = \frac{n_{t+1}}{n_t}$
   * So ρ_t depends only on n_t (which is the sum of n_t^i, but since competitive, n_t^i doesn't influence ρ_t)
   * Amount of stuff the next generation produces, solves: $n_{t+1}u'(n_{t+1}) = g'(n_t)n_t \equiv \nu(n_{t+1}) = G(n_t)$ (This is a non-linear difference equation)
   * Note: there is no M in the above equation. This implies that price levels do not matter - only the ratio of the prices $\rightarrow$ this is an example of the neutrality of money/classical dichotomy
   * Proof of equation above, the CE solution (a difference equation):
   * FOC is $\rho_t u'(\rho_t n_t) = g'(n_t)$
   * subbing MC into FOC; $\rho_t u'(\rho_t n_t) = g'(n_t), \rho_t = \frac{n_{t+1}}{n_t}$
   * $\Rightarrow \frac{n_{t+1}}{n_t} u'\left(\frac{n_{t+1}}{n_t} n_t\right) = g'(n_t)$
   * $\Rightarrow \frac{n_{t+1}}{n_t} u'(n_{t+1}) = g'(n_t)$
   * $\Rightarrow n_{t+1} u'(n_{t+1}) = n_t g'(n_t)$
   * The difference equation will give $\{n_t\}_{t=1}^{\infty}$, use $\pi_t = \frac{M}{n_t}$ to get $\{\pi_t\}_{t=1}^{\infty}$
4. There are lots of equilibria, we'll focus on 2 steady-state equilibria
   (a) Autarky; $n_t = 0, \forall t,\ \pi_t = \infty, \forall t$
      * in any well specified model there should be an equilibrium where money has no value

   (b) $n^* \in (0,1); \rho_t = 1, n_t = n^*, \pi_t = \frac{M}{n^*} \forall t$
      * $u'(n^*) = g'(n^*) \rightarrow$ this is the solution to the planner's problem $\Rightarrow n^* = n^{**}$
      * We are able to decentralize the planner's solution by having fiat money
      * A good example of the fundamental welfare theorem at work
      * Are there other steady states? No, there's a unique, nontrivial steady state besides autarky (only one n^* solves $u'(n^*) = g'(n^*)$)
      * This SS is Pareto superior to autarky - everyone is better off (by our assumptions on u and g, corner sol'n that is autarky is not Pareto optimal.

<u>More on OG Production Economy</u> (NOTE: now we'll get into non-SS solutions)

- Individual optimization (representative generation t, young agents solves):
  – $\max_{0 \le n \le 1} \underbrace{u(\rho_t n)}_{\text{util of cons when old}} - \underbrace{g(n)}_{\text{disutil labor when young}}$, $\rho_t \equiv \frac{\pi_t}{\pi_{t+1}}$ (ratio of price of labor when young to price of goods when old)

- FOC $\Rightarrow \rho_t u'(\underbrace{\rho_t n}_{=c_{t+1}}) = g'(n) \Rightarrow n = \phi(\rho_t)$ (where $\phi(\cdot)$ is the labor supply policy function)

- Market clearing
  - $M = p_t n_t, \forall t$ [$n_t \equiv$ labor supply of rep gen t agent]
  - Substitute for ρ_t using market clearing:
    * find $\rho_t = \frac{\pi_t}{\pi_{t+1}} = \frac{M}{n_t} * \frac{n_{t+1}}{M} = \frac{n_{t+1}}{n_t}$
    * Substitute this into the FOC: $\Rightarrow n_{t+1}u'(n_{t+1}) = n_t g'(n_t)$ (***)
  - A competitive equilibrium is a sequence $\{n_t\}_{t=1}^{\infty}$ which satisfies (***) (i.e., any state that satisfied individual optimization and market clearing)

- AN ASIDE on labor supply:
  - $\frac{\partial n}{\partial \rho_t}$, a change in an endogenous variable (n) for a change in an exogenous variable (ρ_t) - a comparative static
  - Use IFT!
  - Recall, use FOC: $G(n, \rho_t) = \rho_t u'(\rho_t n_t) - g'(n_t) = 0$
  - $\Rightarrow \frac{\partial n}{\partial \rho_t} = \frac{-G_2(n,\rho_t)}{G_1(n,\rho_t)}$
    * $G_1(n, \rho_t) = \rho_t^2 u''(\rho_t n_t) - g''(n_t)$
    * $G_2(n, \rho_t) = \rho_t n u''(\rho_t n) + u'(\rho_t n)$
  - $\Rightarrow \frac{\partial n}{\partial \rho_t} = -\frac{\overbrace{u'(\rho_t n)}^{\text{subs effect}} + \overbrace{\rho_t n u''(\rho_t n)}^{\text{inc effect}}}{\underbrace{\rho_t^2 u''(\rho_t n) - g''(n)}_{(-)}}$ (NOTE: don't know sign of numerator)
  - Numerator: $u'(\rho_t n)\left[1 + \frac{c_{t+1}u''(c_{t+1})}{u'(c_{t+1})}\right], c_{t+1} = \rho_t n$
  - $= u'(c_{t+1})[1 - R(c_{t+1})]$
  - Where $R(x) =$ Coeff of Relative Risk Aversion$\equiv \frac{-xu''(x)}{u'(x)}$
  - $\frac{\partial n}{\partial \rho_t} >< 0$ as $R(c_{t+1}) >< 1$
    * if <1, then gross substitutes case, sign is positive (substitution effect dominates), ($\frac{\partial n}{\partial \rho} > 0$)
    * if >1 then gross complements case, sign is negative(income effect dominates), ($\frac{\partial n}{\partial \rho} < 0$)
    * gross subs case if subs effect dominates and work more as wage increases, gross complements if reverse)
  - Consumption and leisure move in opposite directions
  - substitution effect dominating the income effect is a function of the curvature of the utility function

- Competitive Equilibria (2 types, stationary and not)
  1. Stationary
     (a) $\rho_t = 1 \Rightarrow n_t = n^*$ where $u'(n^*) = g'(n^*), 0 < n^* < 1$
        - DRAW graph showing indiff curve tangent at n^*

     (b) $n_t = 0, \forall t$, Autarky!
  2. Nonstationary Equilibria (non ss, we bifurcate into two cases)
     - Potentially lots
     - Two main groups:
       (a) Gross substitutes ($\frac{\partial n}{\partial \rho_t} > 0$)
       (b) Gross compliments ($\frac{\partial n}{\partial \rho_t} < 0$)

Gross Substitutes

- (***) $n_{t+1}u'(n_{t+1}) = n_t g'(n_t)$ (describes how n_{t+1} depends on n_t)
- put n_1 into (***) and get n_2, put n_2 into (***) and get n_3,...
- DRAW graph with n_t and n_{t+1} as axes. draw 45 degree line. Draw S-curve for difference equation. Note that this is for the gross subs case. Note that along the 45 degree line, we know all points along line are stationary eq. Note how axes are bounded between 0 and 1.
  - If $n_1 < n^*$, then sequence $\{n_t\} \to 0$
  - If $n_1 > n^*$, then sequence $\{n_t\} \to \infty$
  - $\Rightarrow$ 2 steady states, 0 and n^*
  - notice that the the monetary steady state is more fragile/unstable (there is convergence to zero, not n^*)
- $\frac{\partial n_{t+1}}{\partial n_t}|_{n_t=n_{t+1}=n^*} = \frac{g'(n_t)+n_t g''(n_t)}{u'(n_{t+1})+n_{t+1}u''(n_{t+1})}$ (restrict derivative to eval at these points)
  - Properties:
  - positive $\to$ numerator definitely positive , denominator positive because gross subs.
  - $> 1 : g'(n*) = u'(n*)$ and $g''(n^*)$ pos and $n^*u''(n^*)$ neg, so numerator $>$ denominator at SS
    - Means curve crosses 45 degree line from below
    - Implies SS is unstable
  - To see this in G.S. case, note that if $n_1 < n^*$ then $n_2 < n_1$ (b/c $\frac{\partial n_{t+1}}{\partial n_t} > 1$ for $n \leq n^*$ as shown above)
  - Thus, by market clearing, $\frac{n_2}{n_1} = \rho_1 > 1 \Rightarrow n_1 > n_2$ (next generation works less)
  - Thus $n_1 > n_2$ and $\pi_1 < \pi_2$
  - because $M = \pi_t n_t$, as $n_t \to 0$, $\pi_t \to \infty$
    - i.e., get inflation w/o increasing money supply as approach autarkic SS
  - Note that if $n_1 > n^*$, then $n_2 > n_1$ (b/c $\frac{\partial n_{t+1}}{\partial n_t} < 1$ for $n \geq n^*$ as shown above)
  - Thus, by market clearing, $\frac{n_2}{n_1} = \rho_1 < 1 \Rightarrow n_1 < n_2$ (next generation works more)
  - Thus $n_1 < n_2$ and $\pi_1 > \pi_2$
  - because $M = \pi_t n_t$, as $n_t \to \infty$, $\pi_t \to 0$
    - i.e., get deflation w/o constant money supply as labor supply/output expand
- You can pick an n_1 and by (***) it creates a sequence $\{n_t\}_{t=1}^{\infty}$ and this sequence is a competitive equilibrium
- OR
- Pick p_1 (s.t. $0 < n < n^*$) and since $n_1 = \frac{M}{\pi_1}$, we have implicitly picked n_1 and we can then get the equilibrium sequence

Gross Complements

- Draw graph with n_t and n_{t+1} on axes. Have 45 degree line and line crossing that which represents the difference equation.
  - you can pick an n_1 such that the sequence explodes ("webs") down to n^* or jumps between two levels

- whether converge to n^* depends on $\frac{\partial n_{t+1}}{\partial n_t}$

- B/c gross compliments, $\frac{\partial n_{t+1}}{\partial n_t} < 0$, hence downward slope

- Whether the system is stable (tends to stay at n^*) or not, depends on $\frac{\partial n_{t+1}}{\partial n_t}|_{n^*} >< -1$
  - If > -1, then stable
  - If < -1 unstable
  - If $= -1$, then endogenous cycle - jumps back and forth between two n's
    * An example of $\frac{\partial n_{t+1}}{\partial n_t} = -1$ is:
    * $n_{t+1} = \frac{k^2}{n_t} \rightarrow$ a difference equation ($\frac{\partial n_{t+1}}{\partial n_t} = \frac{-k^2}{n_t^2}$ - eval at $n^* = k$...)
    * $\Rightarrow n^* = k$
    * What $u(\cdot)$ and $g(\cdot)$ to get this?
    * e.g. $u(n) = \frac{n^2}{2k}$, $g(n) = kln(n)$

Gale's OG Model

- Generalization of Shell's model

- Environment:
  - Generation of size N_t born in period $t = 1, 2, ...$
  - $N_{t+1} = \gamma N_t \Rightarrow \gamma =$ rate of population growth
  - 2-period lived agents. Identical within a generation.
  - Endowments: $e = (e_0, e_1)$, no production
  - Consumption of generation t: $c(t) = (c_0(t)c_1(t+1))$
  - $u(c)$ increasing and strictly quasi concave

- Feasible Allocations:
  - $N_t e_0 + N_{t-1}e_1 \geq N_t c_0(t) + N_{t-1}c_1(t)$
  - $\Rightarrow \gamma(\underbrace{e_0 - c_0(t)}_{\text{young save}}) + (\underbrace{e_1 - c_1(t)}_{\text{old save}}) \geq 0$ (constraint is within perod t, but between generations of agents)

- Competitive Equilibrium
  - Budget constraint (of generation t agent):
  - let $\rho_t = \frac{\pi_t}{\pi_{t+1}}$, $\rho_t(e_0 - c_0(t)) + e_1 - c_1(t+1) \geq 0$ (this constraint is within a generation, but between periods)
  - Sub B.C. into $u(\cdot, \cdot)$:
    * $\implies u(c_0(t), c_1(t)) = u(c_0(t), \rho_t(e_0 - c_0(t)) + e_1)$
    * $\implies$ FOC (now only 1 b/c just one choice):
    * $\frac{\partial u(\cdot,\cdot)}{\partial c_0(t)} = u_1(\cdot, \cdot) - \rho_t u_2(\cdot, \cdot) = 0$
    * $\implies u_1(\cdot, \cdot) = \rho_t u_2(\cdot, \cdot) \implies \underbrace{\frac{u_1(\cdot, \cdot)}{u_2(\cdot, \cdot)}}_{MRS} = \underbrace{\rho_t}_{\text{slope of B.C.}}$
  - DRAW box with generations on horizontal, periods on the vertical. Show arrow going horizontal that represents feasibility constraint. Show arrow doing down in the vertical that represents the budget constraint

- <u>Steady State Allocations</u> (let's look at these b/c more easy to see)
  - $c(t) = c^*, \forall t$
  - $c_0(t) = c_0^*, c_1(t) = c_1^*, \forall t$
  - $\rho_t = \rho, \forall t$
  - writing the feasible allocation and budget constraint in the steady state:
    * $\gamma(e_0 - c_0^*) + e_1 - c_1^* = 0$ (feasible allocation)
    * $\rho(e_0 - c_0^*) + e_1 - c_1^* = 0$ (Satisfied budget constraint)
  - Putting the two above together, one gets: $(\rho - \gamma)(e_0 - c_0^*) = 0$
  - All SS satisfy this.
  - There are two steady states (i.e., 2 ways to make the above equation =0):
    1. $e_0 = c_0^* \Rightarrow e_1 = c_1^*$, Autarky
    2. $\rho = \gamma$, nontrivial steady state, where prices decline with pop growth
  - Is autarky Pareto optimal?
    * 2 key parameters: γ and $\tilde{\rho}$
    * feasibility, $\gamma(e_0 - c_0^*) + e_1 + c_1^* = 0$
    * $\tilde{\rho}$, parametrizes MRS at e
      · Rate of substitution between periods (see this from FOC above)
    * $\tilde{\rho}$ and γ both exogenous
      · $\tilde{\rho}$ depends on $u(\cdot)$ and endowments
      · $\tilde{\rho} = MRS_e = \frac{u_1(e_0,e_1)}{u_2(e_0,e_1)}$=slope of indiff curve at endowment point)
      · γ is just given
    * $\rho =$ the real interest rate, and is endogenous
    * 2 cases
      1. Classical case
         · $\tilde{\rho} > \gamma$, by assumption
         · $\Rightarrow$ implies budget constraint steeper than feasibility constraint
         · DRAW graph with Old age on vertical (cons and endow) and young on horizontal (cons and endow). Draw in feasibility constraint and budget constraint. Draw indiff curve tangent to constraints such that it hits the feasibility constraint (which has less slope).
         · At $\rho = \gamma, c_0^* > e_0$
         · IS this pareto optimal? No.
         · There is a price at which want to trade, but these leads to infeasible allocations. See this by:
         · Gen 1 young give up ε_1 to initial old
         · Gen 2 young give up consumption to Gen 1 old at a rate of $\tilde{\rho}\varepsilon_1$ for each in Gen 1 old
         · $\Rightarrow$ each Gen 2 gives up: $\varepsilon_2 = \frac{N_1 \tilde{\rho} \varepsilon_1}{N_2} = \frac{\tilde{\rho}}{\gamma}\varepsilon_1$ (b/c $\gamma < \tilde{\rho}$ by assumption)
         · $\implies \varepsilon_2 > \varepsilon_1$
         · $\Rightarrow \varepsilon_{t+1} > \varepsilon_t \Rightarrow$ run out of resources at some point, so not Pareto optimal with trade
         · Autraky is Pareto optimal - can't dominate with feasible trade.
      2. Samuelson case
         · $\tilde{\rho} < \gamma$, by assumption
         · $\Rightarrow$ implies feasibility constraint steeper than budget constraint
         · DRAW graph with Old age on vertical (cons and endow) and young on horizontal (cons and endow). Draw in feasibility constraint and budget constraint. Draw indiff curve tangent to constraints such that it hits the feasibility constraint (which has a steeper slope).

- At $\rho = \gamma, c_0^* < e_0$ - so redistribute from young to old increase utility
- e is not Pareto optimal
- In both cases, c^* is on a higher indifference curve, but only in the Samuelson case can trade support c^*
- e.g. if Gen 1 gives up ε_1 to the initial old, then Gen 2 must give up $\varepsilon_2 = \frac{N_1 \tilde{\rho} \varepsilon_1}{N_2} = \frac{\tilde{\rho}}{\gamma} \varepsilon_1 < \varepsilon_1$ to generation 1.
- i.e., $\varepsilon_{t+1} < \varepsilon_t$
- As t goes to infinity, ε_t goes to zero, so this is feasible.
- Key is that MRS (how trade consumption when young for consumption when old) is smaller than rate of population growth.
- c^* Pareto dominates e
- Can decentralize with money

* So is Autarky Pareto Optimal?
  - Samuelson case? - no
  - Classical case? - yes

OG with Savings

So far we haven't used savings, but problem very similar to that with production:

OG Model w/ Savings

- Gen t:
- $u(c_0(t)) + \nu(c_1(t+1))$ OR $u(e_0 - s(t)) + \nu(e_1 + \rho(t)s(t))$, where $\rho(t) = \frac{\pi_t}{\pi_{t+1}}$ and $s(t)$= savings of Gen t
- Market clearing says: $M = \pi_t s(t),\ t = 1, 2, ...$

OG Model with Production

- Gen t
- $u(\rho_t n_t) - g(n_t)$, where $\rho_t = \frac{\pi_t}{\pi_{t+1}}$
- Market clearing says: $M = \pi_t n_t,\ t = 1, 2...$

Review of some key points:

- Budget constraint $\rightarrow$ for an individual agent
  - Can be nominal or real (i.e., in terms of dollars or goods)
  - e.g. $e = (1, 0)$:
  - real B.C. (in terms of period 1 goods) is: $p_t c_t^t + p_{t+1} c_{t+1}^t = p_t 1 + p_{t+1} 0$
  - where $p_t = \frac{\text{\#\#period 1 goods}}{\text{period t good}}$ OR
  - nominal B.C. (in terms of \$s) is : $\pi_t c_t^t + \pi_{t+1} c_{t+1}^t = \pi_t 1 + \pi_{t+1} 0$
  - where $\pi_t = \frac{\#\$s}{\text{t period good}}$
- Similarly, market clearing can be written in terms of goods or money market
  - Goods: $c_t^t + c_t^{t-1} = e_t^t + e_t^{t-1}$
  - Money: $\pi_t (c_t^t + c_t^{t-1}) = M_t$
- Two solve for equilibrium, 3 steps
  1. Find FOC (agents optimize taking prices as given)
  2. Solve market clearing (MC) to put nominal variables in terms of real variables

3. Plug MC into FOC (prices move to clear markets, given agents decision rules)

- There were 3 ways for agents to move consumption across time that we considered
  1. Storage
     - Can store goods from one period to next
     - May get rate of return on storage (possibly negative) (e.g. PS5, #1b)
  2. Period 0 claims market
     - All generations meet at time 0
     - Trade claims to goods at time t, $p_t = \frac{\text{\# period 1 goods}}{\text{period t good}}$
  3. Fiat money and sequential markets
     - Initial old endowed with money supply
     - Only interact with generations immediately before/after own
     - Money may have value: $\pi_t = \frac{\#\$s}{\text{t period good}}$,